Known : $H^*(ko) = \mathcal{A} /\!/ \mathcal{A}(1) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2$

$\mathcal{A}(1)$ = submodule of Steenrod alg gen. by Sq^1. Sq^2

$\mathcal{A}(1)_* = \mathcal{A}_* / (\bar{\xi}_1^4 . \bar{\xi}_2^2 . \bar{\xi}_3 . \bar{\xi}_4 . \dots)$

$= \mathbb{F}_2 \{ h_{1,0} . h_{1,1} . h_{2,0} \}$.

where $h_{i,j} = \xi_i^{2^j} \in E_1^{1, (2^i-1)2^j, 2i-1}$

Tools :

1. Adams SS : $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^* ko . \mathbb{F}_2) \Rightarrow \pi_{t-s} ko \otimes \mathbb{Z}_2$

$d_r : E_r^{s,t} \longrightarrow E_r^{s+r . t+r-1}$

2. Cotensor : $M . N$ right / left Γ - comodules . Γ flat over A.

$(A . \Gamma)$ Hopf algebroid / field k. $\psi_M : M \longrightarrow \Gamma \otimes_A M$

left A - linear, counital . coassoc. , similarly ψ_N comodule str

on N. Then cotensor is given by

$$0 \longrightarrow M \,\square_\Gamma\, N \longrightarrow M \otimes_A N \xrightarrow{\psi_M \otimes N - M \otimes \psi_N} M \otimes_A \Gamma \otimes_A N.$$

and $M \,\square_\Gamma\, N = N \,\square_\Gamma\, M$ (if defined).

3. Reformulate Adams E_2 - page :

<u>FACT 1</u> $M . N$ left Γ - comods. M proj. / A. then $\mathrm{Hom}_A(M . A)$ is a right Γ - comodule . and

$$\mathrm{Hom}_\Gamma(M . N) = \mathrm{Hom}_A(M . A) \,\square_\Gamma\, N$$

$$\Rightarrow \mathrm{Hom}_\Gamma(A . N) = A \,\square_\Gamma\, N$$

<u>FACT 2</u> M left Γ - comod . $\mathrm{Ext}_\Gamma^i(M . -) = i^{th}$ right derived functor of $\mathrm{Hom}_\Gamma(M . -)$

N right Γ-comod. $\mathrm{Cotor}^i_\Gamma(N, -) = i^{th}$ right derived functor of $N \square_\Gamma -$

<u>FACT 3</u> (Change-of-rings) $f: (k, \Gamma) \longrightarrow (k, \Sigma)$ surj map of Hopf algs, then $\forall$ left Σ-comodule,

$$\mathrm{Ext}_\Gamma(k, \Gamma \square_\Sigma N) = \mathrm{Ext}_\Sigma(k, N)$$

▶ Reformulate:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^* ko, \mathbb{F}_2)$$

$$\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2, H_* ko) \qquad \text{duality between comod / mod.}$$

Since $H^* ko = \mathcal{A} /\!/ \mathcal{A}(1) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2$.

$H_* ko = \mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2$ duality ..

So
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2, H_* ko)$$

$$= \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2)$$

$$= \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2) \qquad \text{Change-of-rings.}$$

$$= \mathrm{Cotor}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$$

$$\Longrightarrow \pi_{t-s} ko \otimes \mathbb{Z}_2.$$

4. May SS: $E_1^{*,*,*} = \mathbb{F}_2[h_{i,j} : i \geq 1, j \geq 0] \Longrightarrow \mathrm{Cotor}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$

w/ $d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1, t, u-r}$

For submodule of $\mathcal{A}_*$, one has, for $\mathcal{A}(1)_*$,

$$E_1^{*,*,*} = \mathbb{F}_2[h_{i,j} : 1 \leq i+j \leq 2] \quad \text{induced by} \quad \mathcal{A}_* \longrightarrow \mathcal{A}(1)_*$$

5. Cobar complex: (Probably not used)

(A, Γ) Hopf algebroid. M, N left / right Γ-comodules.

N proj. / A, then the cobar cpx $C^s_\Gamma(N,M) = N \otimes_A \bar{\Gamma}^s \otimes_A M$

for $\bar{\Gamma} = \operatorname{coker}(\eta_L : A \to \Gamma)$. s.t.

$$d_s : C^s_\Gamma(N.M) \longrightarrow C^{s+1}_\Gamma(N.M)$$

$$x \otimes a_1 \otimes \cdots \otimes a_s \otimes m \mapsto \psi_N(x) \otimes a_1 \otimes \cdots \otimes a_s \otimes m$$
$$+ \sum_{i=1}^{s} (-1)^i x \otimes a_1 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_s \otimes m$$
$$+ (-1)^{s+1} x \otimes a_1 \otimes \cdots \otimes a_s \otimes \psi_M(m).$$

<u>FACT</u> $H^s(C^*_\Gamma(N.M)) = \operatorname{Cotor}^s_\Gamma(N.M)$.

6. FACT : $\mathcal{A}(n)_* = \mathbb{F}_2[\xi_i : i = 1.2.\ldots, n+1] / (\xi_i^{2^{n+2-i}})$

$\mathcal{A}_* = \mathbb{F}_2[\xi_i : i \geq 1]$. $\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$.

Computation :

Note $E_1^{*,*,*} = \mathbb{F}_2[h_{1.0} . h_{1.1} . h_{2.0}]$.

d_1 - cycles are $h_{1.0}$. $h_{1.1}$. $h_{2.0}^2$ (by derivation)

While $d_1(h_{2.0}) = h_{1.0} h_{1.1}$. b/c

- <u>Prp</u> $d_1(h_{i.j}) = \sum_{0<k<i} h_{i-k.\, k+j} h_{k.j}$

pf. $h_{i.j} = \xi_i^{2^j} \in E^0_{2i-1} C_{\mathcal{A}(1)_*}(\mathbb{F}_2 . \mathbb{F}_2)$. where

$E^0_s A = F_s A / F_{s-1} A$ graded piece. and in cobar cpx

$$\delta(\xi_i^{2^j}) = \sum_{k=1}^{i-1} \xi_{i-k}^{2^{k+j}} \otimes \xi_k^{2^j}$$

$$\Rightarrow d_1(h_{i.j}) = \sum_{0<k<i} h_{i-k.\, k+j} h_{k.j}.$$

Thus. by Prp. one has $d_1(h_{2.0}) = h_{1.0} h_{1.1}$.

On the other hand. using Massey product.

$$\langle h_{1.0}.\ h_{1.1}.\ h_{1.0}.\ h_{1.1} \rangle = h_{2.0}^2$$

$$h_{2.0} \quad h_{2.0} \quad h_{2.0}$$

$$0 \quad 0$$

On E_2-page, it's gen. by $h_{1.0}$. $h_{2.0}^2$. $h_{1.1}$. subject to relations $h_{1.0} h_{1.1}$. Now $d_2 : E_2^{s.t.u} \longrightarrow E_2^{s+1.t.u-2}$.

$h_{1.0} \in E_2^{1.1.1}$. $h_{1.1} \in E_2^{1.2.1}$. $h_{2.0} \in E_1^{1.3.3}$. $h_{2.0}^2 \in E_2^{2.6.6}$.

By degree reason. no non-trivial d_2. So $E_3 = E_2$.

In fact. even pages don't have non-trivial differentials. $E_{2n} = E_{2n+1}$ (first grading & 3rd grading have same odevity).

On E_3-page. it's gen. by $h_{1.0}$. $h_{1.1}$. $h_{2.0}^2$. To compute the differentials. Note $d_3(h_{1.0}) = d_3(h_{1.1}) = 0$.

- <u>Prop</u> $d_r(h_{1.j}) = 0$. $\forall r.j$.

  pf. $h_{1.j} = \xi_1^{2^j} \in E_1^0\, C_{A(1)_*}(\mathbb{F}_2.\ \mathbb{F}_2)$

  $$S(\xi_1^{2^j}) = \sum_{k=1}^{1-1} \cdots = 0$$

Suffice to compute $d_3(h_{2.0}^2) = d_3(\langle h_{1.0}.\ h_{1.1}.\ h_{1.0}.\ h_{1.1} \rangle)$.

To use higher Leibniz rule.

- <u>Thm</u> (Higher Leibniz rule. May 1969)

Let C be a dga w/ increasing filtration w/ inducing SS indexed s.t. $d_r : E_r^{s,t} \longrightarrow E_r^{s+1,t-r}$. If $\langle x_1, \dots, x_n \rangle$ defined in E_{r+1} w/ each x_i matrix w/ entries being permanent cycles, $x_i \longrightarrow \beta_i$ in MH^*C. Let k be w/ $1 \le k \le n-2$ s.t. $\langle \beta_i, \dots, \beta_{i+k} \rangle$ strictly defined in H^*C, and that each entry of $a_{i,j}$ w/ $1 < j-i \le k$ in the defining system for $\langle x_1, \dots, x_n \rangle$ has bidegree (p,q), then each elet of $E_{r+m+1}^{p,q+m}$ w/ $m \ge 0$ is a permanent cycle. Let $s > r$ be s.t. each (p,q) as above w/ $k < j-i < n$ and for each t w/ $r < t < s$, $E_t^{p+1,q-t} = 0$, and if $j-i > k+1$, then $E_{r+s-t}^{p+1,q-t} = 0$. Then for each $x \in \langle x_1, \dots, x_n \rangle$

$$d_t(x) = 0, \quad \forall \; r < t < s.$$

Besides, there are permanent cycles $\delta_i \in ME_{r+1}$ for $1 \le i \le n-k$ converging to elets in $\langle \beta_i, \dots, \beta_{i+k} \rangle$ s.t. $\langle \gamma_1, \dots, \gamma_{n-k} \rangle$ is defined in E_{r+1}, and contains an elet γ surviving to $d_s(x)$, where

$$\gamma_1 = \begin{pmatrix} \delta_1 & x_1 \end{pmatrix}$$

$$\gamma_i = \begin{pmatrix} x_{i+k} & 0 \\ \delta_i & x_i \end{pmatrix} \qquad 1 < i < n-k$$

$$\gamma_{n-k} = \begin{pmatrix} x_n \\ \delta_{n-k} \end{pmatrix}$$

Assume further that each δ_i is unique, that each $\langle x_1, \dots, x_{i-1}, \delta_i, x_{i+k+1}, \dots, x_n \rangle$ is strictly defined, and all Massey products in sight, except for possibly $\langle \beta_i, \dots, \beta_{i+k} \rangle$ have 0

indeterminacy, then we have

$$d_s(\langle x_1, \dots, x_n \rangle) = \sum_{i=1}^{n-k} \langle x_1, \dots, x_{i-1}, \delta_i, x_{i+k+1}, \dots, x_n \rangle .$$

In our case, $s=3$, $r=1$ ($\langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle = h_{2,0}^2$ is defined in E_2-page), $n=4$, $k=2$ (since $\langle h_{1,0}, h_{1,1}, h_{1,0} \rangle$ and $\langle h_{1,1}, h_{1,0}, h_{1,1} \rangle$ strictly defined). Thus

$$\begin{aligned} d_3(\langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle) &= \langle \delta_1, h_{1,1} \rangle \\ &\quad + \langle h_{1,0}, \delta_2 \rangle \\ &= \delta_1 \cdot h_{1,1} + h_{1,0} \cdot \delta_2 \end{aligned}$$

where $\delta_1 \in \langle h_{1,0}, h_{1,1}, h_{1,0} \rangle = h_{1,1}^2$

$\delta_2 \in \langle h_{1,1}, h_{1,0}, h_{1,1} \rangle = h_{1,0} h_{1,2}$

- FACT $\langle h_{1,j}, x, h_{1,j} \rangle = h_{1,j+1}\, x$.

Thus $d_3 h_{2,0}^2 = h_{1,1}^3 + h_{1,0}^2 h_{1,2}$.

Now, we get our E_3. Move to E_4. The generators of E_4 are $h_{1,0}$, $h_{1,1}$, $h_{2,0}^4$. Can check

$$\langle h_{1,0}, h_{1,1}, h_{1,1}^2 \rangle = h_{1,0} h_{2,0}^2$$

$$\langle h_{1,1}, h_{1,1}^2, h_{1,1}, h_{1,1}^2 \rangle = h_{2,0}^4$$

$h_{2,0}^4 \in E_4^{4,12,12}$. By degree reason, $E_4 = E_\infty$. No non-trivial diffs on E_4-page.

To get the correct ring structure and know to what elet each elet is mapped to. we need the following theorem:

- <u>Thm</u> (May convergence theorem)

  With notions given in higher Leibniz rule. let $\langle x_1, \dots, x_n \rangle$ be defined in E_{r+1}. x_i matrix w/ entries being permanent cycles and $x_i \longrightarrow \beta_i \in MH^*C$. If $\langle \beta_1, \dots, \beta_n \rangle$ strictly defined and there are no crossing diffs (i.e. if an entry of $a_{i,j}$ w/ $1 < j-i < n$ in the defining system for $\langle x_1, \dots, x_n \rangle$ has bideg (p, q). then each elet of $E_{r+m+1}^{p, q+m}$ w/ $m \geq 0$ is a permanent cycle). Then each elet in $\langle x_1, \dots, x_n \rangle$ is a permanent cycle converging to an elet in $\langle \beta_1, \dots \beta_n \rangle$.

- <u>Thm</u> (Moss convergence theorem)

  Suppose x_1, x_2, x_3 be permanent cycles s.t. $x_1 x_2 = x_2 x_3 = 0$ Let x_1, x_2, x_3 be realized in E_∞ by htpy classes $\alpha_1, \alpha_2, \alpha_3$. s.t. $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$. Also there is no crossing diffs for $x_1 x_2$, $x_2 x_3$. Then there is a permanent cycle $e \in \langle x_1, x_2, x_3 \rangle$ that is realized in E_∞ by an elet of Toda bracket $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

Now denote $b_{2,0} := h_{2,0}^2$. Adams SS E_2-page $= E_\infty$ for degree reasons (no non-trivial diffs). Thus, one has

$$\pi_* ko = \hat{\mathbb{Z}_2}[\alpha . \beta . \eta] / \text{Relations}$$

where $h_{1,0} \longrightarrow 2$ (multiplication by 2)

$h_{1,1} \longrightarrow \eta$

$h_{1,0} b_{2,0} \longrightarrow \alpha \leftrightsquigarrow \langle h_{1,0} . h_{1,1} . h_{1,1}^2 \rangle$

$b_{2,0}^2 \longrightarrow \beta \leftrightsquigarrow \langle h_{1,1} . h_{1,1}^2 . h_{1,1} . h_{1,1}^2 \rangle .$

So $\alpha = \langle 2 . \eta . \eta^2 \rangle$. $\beta = \langle \eta . \eta^2 . \eta . \eta^2 \rangle$ as Toda brackets. and Relations are given by

$$d_1 h_{2,0} = h_{1,0} h_{1,1} \rightsquigarrow 2\eta = 0$$

$$d_3 h_{2,0}^2 = h_{1,1}^3 + h_{1,0}^2 h_{1,2} \rightsquigarrow \eta^3 + 4h_{1,2} = 0$$

but $h_{1,2} = 0 \Rightarrow \eta^3 = 0$

Since $h_{1,0} h_{1,1}$ not survive, we have to ask $\alpha\eta = 0$.

By construction. we also want $\alpha^2 - 4\beta = 0$ since $h_{1,0}^2 \longrightarrow 4$

Thus. we obtain

$$\pi_* ko = \hat{\mathbb{Z}_2}[\alpha . \beta . \eta] / (\eta^3 . 2\eta . \alpha^2 - 4\beta . \alpha\eta).$$